

# Charge fractionization in N=2 supersymmetric QCD

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It is shown that the physical “quark number” charges which appear in the central charge of the supersymmetry algebra of  $N = 2$  supersymmetric QCD can take irrational values and depend non trivially on the Higgs expectation value. This gives a physical interpretation of the constant shifts which the “electric” and “magnetic” variables  $a_D$  and  $a$  undergo when encircling a singularity, and show that duality in this model is truly an electric-magnetic-quark number duality. Also included is a computation of the monodromy matrices directly in the microscopic theory.

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During the last couple of years, huge progress has been made in the understanding of four dimensional  $N = 2$  supersymmetric gauge theories, following the work of Seiberg and Witten [1,2], where the low energy (wilsonian) effective action was computed exactly up to two derivatives and four fermions terms for the gauge group SU(2). The mathematical structure that emerged there was then generalized to obtain very plausible solutions for general gauge groups  $G$  (see [3] and [4] with references therein). All these theories have a complex manifold of inequivalent vacua (moduli space). This degeneracy comes from flat directions in the scalar potential which cannot be lifted quantum mechanically due to tight constraints imposed by  $N = 2$  supersymmetry [5]. The moduli space has a Coulomb branch where the gauge group, of rank  $r$ , is spontaneously broken down to  $U(1)^r$ . Many interesting physical phenomena occur along the Coulomb branch, including the appearance of massless solitonic states at strong coupling [1,2], a subtle realization of electric-magnetic duality at the level of the low energy physics [1,2], the existence of non-trivial Conformal Field Theory in four dimensions [6] and discontinuities in the spectrum of stable BPS states [7]. These “exactly soluble” theories are likely to play an outstanding rôle in our understanding of more realistic gauge theories as QCD, like the Ising model did for critical phenomena.

In this letter is explained how to compute from first principles some abelian charges appearing in the central charge  $Z$  of the supersymmetry algebra. Then, knowing  $Z$ , one can compute the mass  $m$  of any state lying in a small representation of the supersymmetry algebra (eight dimensional for a CPT conjugate multiplet):

$$m = \sqrt{2} |Z|. \quad (1)$$

All the elementary excitations (quarks, W bosons, . . .),

as well as the known solitonic states (monopoles and dyons) are BPS states and their masses are thus given by (1). We will also obtain an interesting physical interpretation of curious monodromy properties, and we will be able to compute the monodromy matrices directly in the microscopic theory. This study will finally solve some puzzles about the BPS mass formula and the renormalization group flow.

## I. PRESENTATION OF THE PROBLEM

Along the Coulomb branch, the central charge  $Z$  of  $N = 2$  supersymmetric QCD contains, in addition to the electric and magnetic charges  $Q_e$  and  $Q_m$ , the “quark number” charges  $S_f$ . These charges correspond to the invariance of the lagrangian under the transformations  $Q_f \mapsto e^{i\alpha} Q_f$ ,  $\tilde{Q}_f \mapsto e^{-i\alpha} \tilde{Q}_f$ , where  $Q_f$  and  $\tilde{Q}_f$  are the  $N = 1$  chiral superfields making up a matter  $N = 2$  hypermultiplet. Classically  $Z$  reads

$$Z_{cl} = 2a \left( \frac{1}{g} Q_e + \frac{i}{g} Q_m \right) + \frac{1}{\sqrt{2}} \sum_{f=1}^{N_f} m_f S_f, \quad (2)$$

where  $N_f$  is the number of flavours,  $m_f$  the bare mass of the hypermultiplet  $(Q_f, \tilde{Q}_f)$ , and  $g$  the gauge coupling constant. For the gauge group SU(2), on which I will focus for conciseness, the Dirac quantization condition can be written  $Q_m = 4\pi n_m/g$ , where  $n_m$  is an integer also called the “magnetic charge.” In [2], an exact quantum formula for  $Z$  was proposed:  $Z = a n_e + a_D n_m + \frac{1}{\sqrt{2}} \sum_f m_f S_f$ . Here  $a$  is the Higgs expectation value,  $\langle \phi \rangle = a \sigma_3$ , and  $a_D = \frac{1}{2} \partial_a \mathcal{F}(a)$  is the dual variable which can be expressed in terms of the prepotential  $\mathcal{F}$  governing the low energy effective action. This

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formula for  $Z$  is a straightforward generalization of the corresponding formula for the pure gauge theory derived in [1] using electric-magnetic duality arguments. However, we will see that interpreting  $S_f$  in this formula as being the physical quark number is not free from contradictions. Problems also arise when considering that the term  $an_e + a_D n_m$  stems from corrections to the physical electric and magnetic charges alone.

To understand the origin of these difficulties, let us set for a moment the bare masses to zero. In this case the quantum formula  $Z = an_e + a_D n_m$  is not obtained from (2) by simply replacing  $g$  by the running coupling constant, even at one loop. One also has to take into account that, because CP invariance is spontaneously broken by  $\Im m a \neq 0$ , the physical electric charge can pick up terms in addition to  $gn_e/2$ . The simplest example of this phenomenon was first studied by Witten in [8]. In the theories with zero bare masses, all CP violation can be absorbed in a  $\theta$  angle by performing a chiral  $U(1)_R$  transformation. The formula of [8] can then be readily adapted to our case and yields  $Q_e = \frac{g}{2}(n_e - \frac{4-N_f}{\pi}n_m \arg a)$ . Note that the  $S_f$  charges are not affected by a  $\theta$  term since the latter appears in the lagrangian in front of  $F\tilde{F}$  which does not transform under  $S_f$ . Thus when  $m_f = 0$ ,  $S_f$  is expected to have the value one can compute in the CP conserving theories. For instance,  $S_f = \pm 1/2$  when  $n_m = 1$  [9].

However, when the bare masses are non-zero, we would expect  $S_f$  to depend on the Higgs expectation value, or alternatively on the gauge invariant coordinate on the Coulomb branch,  $u = \text{tr } \langle \phi^2 \rangle$ . This is strictly analogous to the phenomenon first discovered in [10], where irrational fermion number values were found in some CP violating field theories. At first sight, this seems bizarre. As  $S_f$  are real numbers, a non-trivial  $u$ -dependence would seem to violate holomorphy. One could then be tempted to forget about CP breaking and argue that, because of supersymmetry, the  $S_f$  charges must be constant and equal to the values one computes in CP conserving theories. But one then faces another, more subtle, difficulty. Suppose one is studying the renormalization group flow, say from the  $N_f = 1$  to the  $N_f = 0$  theory, and that in particular one is trying to deduce the spectrum of stable BPS states of the  $N_f = 0$  theory from the one of the  $N_f = 1$  theory. What should occur is that some states of the  $N_f = 1$  theory, becoming infinitely massive in the process, disappear from the spectrum of the  $N_f = 0$  theory, and that other states, remaining of finite mass, finally constitute the stable BPS states of the  $N_f = 0$  theory. Since the work in [7], we know that the spectrum of the  $N_f = 0$  theory is indeed strictly included in the spectrum of the  $N_f = 1$  theory. Limiting the discussion to the weak-coupling spectra, which can be described semiclassically, all the monopoles of odd  $n_e$  disappear when one goes from  $N_f = 1$  to  $N_f = 0$ , while the monopoles

of even  $n_e$  form the solitonic spectrum of the pure gauge theory. But this is incompatible with the previous formula for  $Z$ . If  $S = \pm 1/2$  for the monopoles, their masses  $m = \sqrt{2}|Z|$  will diverge whatever their electric charge  $n_e$ , since  $a_D$  and  $a$  must flow towards the solution of the pure gauge theory under the action of the renormalization group. So one must definitively give up the idea that the constants  $S_f$  appearing in  $Z$  could be the physical charges. I will rename these constants  $s_f$  and reserve the symbol  $S_f$  for the physical charges. The  $s_f$  may be zero even for monopoles  $n_m = 1$ . The exact quantum formula for the central charge is then

$$Z = an_e + a_D n_m + \frac{1}{\sqrt{2}} \sum_{f=1}^{N_f} m_f s_f. \quad (3)$$

In the remainder of this letter I will explain where the physical charges  $S_f$  hide and how to compute the numbers  $s_f$ . We will also find a physical explanation of the curious monodromy properties  $a_D$  and  $a$  have in the massive theories (they pick up constants in addition to the standard  $SL(2, \mathbf{Z})$  transformations), and a new method to compute the monodromy matrices, directly in the microscopic theory.

## II. THE COMPUTATION OF THE PHYSICAL CHARGES

In this section, I outline the computation of the physical charges  $S_f$  and of the electric charge, focusing on the contribution of the fermions. This yields the most interesting results. A detailed discussion of possible additional contributions, the generalization to any gauge group, as well as the discussion of other physical aspects related to CP invariance in our theories shall be published elsewhere.

To study the semiclassical contributions to the  $S_f$  charge of the Dirac fermions  $\chi_f$  belonging to the hypermultiplet  $Q_f, \tilde{Q}_f$ , we need to quantize the Dirac field around a non-trivial monopole background characterized by a fixed magnetic charge  $n_m$ . The Dirac equation is

$$i\sqrt{2}\gamma^\mu D_\mu \chi_f = (M + i\gamma^5 N) \chi_f + \frac{1}{\sqrt{2}} ((\Re e m_f) \chi_f + i(\Im m m_f) \gamma^5 \chi_f), \quad (4)$$

where  $M = \Re e \phi$  and  $N = \Im m \phi$ . Usually one focuses on the (complex) zero modes of this equation, whose number  $k(n_m)$  is given by an index theorem of Callias [11]. Each zero mode carries one unit of  $S_f$  charge, which shows that if  $-S_{f,m}(n_m)$  is the minimal value  $S_f$  can reach in the monopole sector  $n_m$ , the set of allowed values of  $S_f$  will be  $\{-S_{f,m}(n_m), -S_{f,m}(n_m) + 1, \dots, -S_{f,m}(n_m) + k(n_m)\}$ . If (4) were CP invariant, one would deduce that  $S_{f,m}(n_m) = k(n_m)/2$  because  $S_f$  is odd under CP.

However, in our case we cannot forget about the massive modes of (4). When CP is violated, the density of states having positive energy differs from the density of states having negative energy. This means that the Dirac operator under consideration has a non-zero Atiyah-Patodi-Singer  $\eta$  invariant,<sup>1</sup> which formally reads  $\eta = \sum_{E_n > 0} 1 - \sum_{E_n < 0} 1$  where  $n$  labels the energy levels. It is not difficult to relate the spectral asymmetry quantified by  $\eta$  to the  $S_f$  charge, carefully taking into account the fact that  $S_f$  must be odd under CP. One finds

$$S_{f,m}(n_m) = \frac{1}{2} \eta. \quad (5)$$

Fortunately, the computation of the  $\eta$  invariants of various Dirac operator, and the application to charge fractionization, has been extensively studied in the literature. These works were motivated not only for purely field theoretic reasons, but also because of their important phenomenological applications in the physics of linearly conjugated polymers (see e.g. [12]). In particular, in [13], a very general method, applicable to our Dirac operator, was developed. See also [14], and [15] for a review. The result of the computation is

$$S_{f,m}(n_m) = -\frac{n_m}{2\pi} \arg \frac{a + m_f/\sqrt{2}}{m_f/\sqrt{2} - a}. \quad (6)$$

The same type of technique can be applied to evaluate the physical electric charge, which picks up terms in addition to the standard Witten effect [8] term. One can for instance use the Gauss law, following [16], and find for our theory

$$\frac{2}{g} Q_e = n_e - \frac{4}{\pi} n_m \arg a + \frac{n_m}{2\pi} \sum_{f=1}^{N_f} \arg(m_f^2 - 2a^2). \quad (7)$$

### III. PHYSICAL ANALYSIS

Let us discuss the physical meaning of the formulas found in the previous Section. Note that we have a singular point when  $a = \pm m_f/\sqrt{2}$ , which corresponds to a quark becoming massless. It is very instructive to study the monodromy properties of  $S_f$  around this singularity. Since all the non-trivial,  $u$ -dependent part of  $S_f$  is included in  $-S_{f,m}$ , encircling the singularity at  $a = m_f/\sqrt{2}$  yields  $S_f(u) \mapsto S_f(u) - n_m$ . This is reminiscent of the

shift  $s_f \mapsto s_f + n_m$  the constant  $s_f$  undergoes [2].<sup>2</sup> However, the sign difference between the transformations of  $S_f$  and  $s_f$  is crucial. It definitively proves that  $s_f$  cannot be identified with  $S_f$ . Moreover, it shows that  $S_f(u)$  and  $a_D(u)$  pick up the same term under the monodromy. This simply means that  $S_f(u)$  is already included in  $a_D(u)$  (at weak coupling), and is responsible for the curious constant shift  $a_D$  was known to undergo since the work in [2]. In the strong coupling region, since  $a$  and  $a_D$  are intimately related due to the non-abelian monodromies, the  $S_f$  charges will also contribute to  $a$ . Note however that the distinction between  $Q_e$ ,  $Q_m$  and  $S_f$  is very unclear in the strong coupling region, and that the natural quantities to use are  $a$  and  $a_D$ .

The fact that the variables  $a$  and  $a_D$  do pick up contributions from the electric, magnetic and  $S_f$  charges is very interesting from the physical point of view. This means that duality in the theories with non zero bare masses is really an electric-magnetic- $S_f$  duality! This phenomenon is likely to be quite general when abelian charges appear in the central charge in addition to the electric and magnetic charge.

Now, it should be clear that the variables  $s_f$  appearing in the formula (3) are just constant parts of the  $S_f$  charges not already included in  $a_D$  and  $a$ . In particular one can compute  $s_f$  in the weak coupling region by studying the asymptotics of  $a_D$  when  $u$  and  $m_f$  are large comparing to the dynamically generated scale of the theory, then extract the terms contributing to  $S_f$  from this asymptotics, and choose  $s_f$  in order to match with the formula  $S_f = -S_{f,m}(n_f) + p$ . Here  $p$  is an integer between 0 and  $k(n_f)$ , see Section 2.

Let us close this Section computing the  $SL(2, \mathbf{Z})$  monodromy matrix  $M$  corresponding to a singularity due to a quark becoming massless. This can easily be done using (7) which shows that  $Q_e$  contribute to  $a_D$  through the term  $-\frac{4}{\pi} a \arg a + \frac{1}{2\pi} a \sum \arg(m_f^2 - 2a^2)$ , see (2). Thus, when no bare masses coincide,

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (8)$$

since  $a$ , being a good local coordinate around  $u = m_f^2$ , obviously does not transform. This result agrees with the standard computation from the low energy effective theory. When some bare masses coincide, we obtain the monodromy matrix corresponding to several hypermultiplets becoming massless at the same time. Then, performing  $SL(2, \mathbf{Z})$  transformations, it is possible to de-

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<sup>1</sup>Note that the operator is defined on an open space here, thus we are dealing with a generalized  $\eta$  invariant.

<sup>2</sup>Though constants, the  $s_f$  do transform non trivially when encircling a singularity, in the same sense as the constants  $n_e$  and  $n_m$  are mixed by the monodromy matrix.

duce the most general monodromy matrix corresponding to any number of  $(n_m, n_e)$  states becoming massless.

#### IV. CONCLUSIONS

We gained interesting physical insight in the meaning of duality in  $N = 2$  supersymmetric QCD by using semiclassical methods. This was possible since when bare masses are much larger than the dynamically generated scale of the theory, some singularities can be present at weak coupling. We found that in this regime the contribution of the physical electric charge  $Q_e$  to  $a_D$  can account for the  $SL(2, \mathbf{Z})$  monodromy matrices associated with the weak coupling singularities. This provides a new way to derive these matrices, directly in the microscopic theory. More important, we saw that the physical  $S_f$  charges contribute to  $a_D$  and can account for the constant shift this variable undergoes. At strong coupling, the *three* abelian charges appearing in the central charge of the supersymmetry algebra will intimately mix together, providing an example of an electric-magnetic-quark number duality.

#### NOTE ADDED

After the first appearance of this work on the hep-th archive (9609101), a preprint appeared [17] where the scenario described in Section 1 for the renormalization group flow is shown to occur. For the  $N_f = 1$  theory, the authors of [17] were able to find, within a string theory framework, that  $s = 0$  for the monopole  $(1, 0)$  and  $s = -1$  for the dyon  $(1, 1)$ . This would contradict the semiclassical computation if  $s$  were the physical charge, but is in perfect agreement with the discussion of Section 1.

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